

ON THE ANALYTICAL EXTENSION OF FUNCTIONS DEFINED BY FACTORIAL SERIES

A Dissertation

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY IN THE
UNIVERSITY OF MICHIGAN

BY

HOWARD K. HUGHES

Reprinted from THE AMERICAN JOURNAL OF MATHEMATICS,
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CONTENTS.

	PAGE
Introduction	757
I. Fundamental Theorems in the Theory of Factorial Series .	758
II. Analytical Extension of Functions Defined by Factorial Series	761
Bibliography	779



ON THE ANALYTICAL EXTENSION OF FUNCTIONS DEFINED BY FACTORIAL SERIES.*

By HOWARD K. HUGHES.

1. *Introduction.* In this paper we shall be concerned with the two kinds of infinite series known, respectively, as "factorial series of the first kind" and "factorial series of the second kind." A factorial series of the first kind is a series of the form

$$(1) \quad \sum_{n=0}^{\infty} a_n n! / z(z+1) \cdots (z+n),$$

while a factorial series of the second kind is one of the form

$$(2) \quad a_0 + \sum_{n=1}^{\infty} a_n (z-1)(z-2) \cdots (z-n) / n! = \sum_{n=0}^{\infty} a_n \binom{z-1}{n}.$$

In each case z represents a variable, real or complex, and the coefficients $a_0, a_1, a_2, \dots$ are independent of z .

Series of form (1) are of fundamental importance in the study of the solutions of difference equations, and play a rôle which is analogous to that played by power series in expressing the solutions of differential equations.

On the other hand, series (2) is of less value in function theory since the representation of a given function $f(z)$ in such a series is sometimes possible in more than one way.† However series (2) frequently enters in problems in interpolation, it being seen that the coefficients a_n are equal to the successive differences of the function $f(z)$ at the point $z = 1$.

Factorial series apparently made their first appearance in the work of Newton and Stirling. They were used by Schlömilch about the middle of the nineteenth century in connection with studies concerning definite integrals. Nevertheless, it is not until recent years that a critical study has been made of the series in question, particularly as to their convergence and analytic properties. This has been accomplished in large measure by Bendixson, Nielsen, Landau and Nörlund. A list of papers relative to such investigations will be found in the Bibliography. Still more recently, H. Bohr has discovered some results concerning the summability of series (1). Nörlund has investigated further the possibility of developing a given function $f(z)$ in

* Presented to the American Mathematical Society at Chicago, April 4, 1931.

† See, for example, Nörlund (5), pp. 224-226. (Bibliography at end of article.)

factorial series of both kinds, and has also discovered some noteworthy results on the analytical extension of functions defined by such series. He has employed extensively factorial series of the first kind in determining the solutions of linear difference equations. Moreover, certain generalized forms of series (1) and (2) have been studied by Pincherle, Carmichael, Fort, and others.

The object of this paper is to obtain new results pertaining to the analytical extension of a function such as is defined by series (1) or (2). As has been shown by Nielsen, the region of convergence of a factorial series is, in general, a half-plane, lying to the right of a vertical line in the z complex plane, this line being called the *boundary line of convergence*. Our problem is to extend analytically into regions to the left of the boundary line of convergence the function defined by such a series, the series itself being regarded as given.

Chapter I is expository, and sets forth some of the fundamental facts concerning factorial series as discovered by Nielsen, Landau, and others. We include also a brief discussion of the results already noted by Nörlund on the analytical extension of such series. Chapter II is devoted to results on analytical extension which are believed to be new, these results being embodied in three theorems. As to the methods employed, the proof of Theorem I rests upon the properties of a certain definite integral, while the proofs of Theorems II and III rest upon the calculus of residues. For this reason the chapter has been divided into two parts. Theorem I, including its proof, has been embodied in Part I, while Theorems II and III have been embodied in Part II. As a corollary to Theorem III, certain results pertaining to the asymptotic development of a function $W(z)$ defined by series (2) have been obtained. Several examples have been introduced to illustrate the results.

CHAPTER I.

FUNDAMENTAL THEOREMS IN THE THEORY OF FACTORIAL SERIES.

2. The purpose of this chapter is twofold. In the first place we shall present in concise form several theorems, now classical, concerning the convergence of the two kinds of series mentioned in the Introduction, namely,

$$(1) \quad \sum_{n=0}^{\infty} a_n n! / z(z+1) \cdots (z+n)$$

and

$$(2) \quad \sum_{n=0}^{\infty} a_n \binom{z-1}{n}.$$

These theorems are fundamental in what follows. In the second place we shall indicate briefly the results which have been obtained by Nörlund on the analytical extension of functions such as are defined by series (1) and (2).

The domain of convergence of factorial series has been studied by Nielsen, Pincherle, Landau, and others. Moreover, Nörlund has deduced a noteworthy result concerning the region of uniform convergence. The more important results pertaining to both convergence and uniform convergence, and the consequences thereof, and those concerning analytical extension will now be briefly stated, the proofs, however, being omitted.

I* (a). If series (1) converges for $z = z_0$, then it converges for all values of z for which $R(z) > R(z_0)$ † except at the points $z = 0, -1, -2, \dots$.

(b). If series (2) converges for $z = z_0$, where z_0 is not a positive integer, then it converges for all values of z for which $R(z) > R(z_0)$.

In case the series converges neither everywhere nor nowhere, it follows from the foregoing theorem that the region of convergence of series (1) or (2) is a half-plane, bounded on the left by a line of the form $R(z) = \lambda$, the number λ being called the *abscissa of convergence*. The series converges everywhere to the right of the line $R(z) = \lambda$, and diverges everywhere to the left of the same line. On the line the series may either converge or diverge. In the case of series (1), the points $z = 0, -1, -2, \dots$, should any such points lie to the right of the boundary line of convergence, are excluded from the domain of convergence, since at these points the terms of the series become infinite. Series (2) may also be said to converge for positive integral values of z , should any such points lie to the left of the line $R(z) = \lambda$, since at these points the series reduces to a polynomial. Theorems will be stated presently showing how λ may actually be determined for any given series of form (1) or (2).

Landau has proved the following theorem concerning the uniform convergence of series (1) and (2):

II. Each of the two series (1) and (2) converges uniformly in any region R which, together with its boundary, lies interior to the half-plane of convergence of the series, provided in the case of series (1) that R does not include any of the points $z = 0, -1, -2, \dots$.

Moreover Nörlund has shown that if series (1) or (2) converges at $z = z_0$ (in case of series (2), $z_0 \neq$ positive integer), then it converges uniformly in any sector δ whose vertex is at z_0 , and which is characterized by the inequalities

* See Nielsen (2), pp. 416-429; (8), p. 124, 235; also Bendixson (1), pp. 15-20.

† Whenever the symbol $R(z)$ occurs, it is understood to mean the *real part of z* .

$$0 \leq |z - z_0| \leq G, \quad -\pi/2 + \epsilon \leq \arg(z - z_0) \leq \pi/2 - \epsilon,$$

where G is any positive number, and ϵ is an arbitrarily small positive quantity. In case of series (1), the neighborhoods of the points $z = 0, -1, -2, \dots$, should any such points lie in δ , are excepted.

As an immediate consequence of II, we have

III. Each of the two series (1) and (2) represents an analytic function in its half-plane of convergence, the neighborhoods of the points $z = 0, -1, -2, \dots$ being excepted in the case of series (1).

We next note the manner in which the actual value of λ pertaining to any given series (1) or (2) may be determined. For this purpose, let us set

$$\alpha = \overline{\lim}_{n \rightarrow \infty} \log \left| \sum_0^n a_n \right| / \log n, \quad \beta = \overline{\lim}_{n \rightarrow \infty} \log \left| \sum_0^{n-1} (-1)^n a_n \right| / \log n,$$

$$\alpha' = \overline{\lim}_{n \rightarrow \infty} \log \left| \sum_{n+1}^\infty a_n \right| / \log n, \quad \beta' = \overline{\lim}_{n \rightarrow \infty} \log \left| \sum_n^\infty (-1)^n a_n \right| / \log n.$$

Then we have the following theorem: *

IV. The abscissa λ of convergence of series (1) is given by α or α' according as the series $\sum_{n=0}^\infty a_n$ is divergent or convergent; and the abscissa λ of convergence of series (2) is given by β or β' according as the series itself is divergent or convergent for $z = 0$.

As limiting cases we may have $\lambda = -\infty$ or $\lambda = +\infty$, corresponding to which cases the series converges or diverges respectively for all values of z .

3. We shall now mention briefly some results obtained by Nörlund on the analytical extension of functions such as are defined by series (1) and (2). We have the following four theorems: †

V. Any function $\Omega(z)$ which can be developed in a series of form (1) can also be developed in the form

$$(3) \quad \Omega(z) = \sum_{n=0}^\infty c_n n! / (z + \omega)(z + 2\omega) \cdots (z + n\omega)$$

where ω is any given constant. Let λ_ω denote the corresponding abscissa of convergence. Moreover let ω_1 be any fixed value of ω such that $\omega_1 > 1$, and let ω' be any variable value such that $\omega' > \omega_1$. Then $\lim_{\omega' \rightarrow \infty} \lambda_{\omega'}$ exists, and is such that

$$\lambda_{\omega'} \leq \lambda_{\omega_1} \leq \lambda.$$

* Landau (1), p. 156, 195; Nörlund (5), p. 223, 257.

† Nörlund (1); (2), pp. 341-379; (5), pp. 233-240, and 262-266.

As an immediate consequence of V, we have the following:

VI. Series (3) when the (variable) quantity ω is real and greater than unity, furnishes the analytical extension of the function $\Omega(z)$ defined by (1) throughout the strip

$$\lambda_{\omega} < R(z) \leq \lambda$$

except in the neighborhoods of the points $z = 0, -1, -2$, etc.

VII. Any function $W(z)$ which can be developed in a series of form (2) can also be developed in the form

$$(4) \quad W(z) = \sum_{n=0}^{\infty} c_n (z - \omega) (z - 2\omega) \cdots (z - n\omega) / n!$$

where ω is any given constant. Let λ_{ω} represent the corresponding abscissa of convergence. Moreover let ω_1 be any fixed value of ω , and ω' any variable value such that $0 < \omega' < \omega_1 < 1$, then $\lim_{\omega' \rightarrow 0} \lambda_{\omega'}$ exists and is such that

$$\lambda_{\omega'} \leq \lambda_{\omega_1} \leq \lambda.$$

Hence we have, corresponding to VI above, the following:

VIII. Series (4), when the (variable) quantity ω is real and such that $0 < \omega < 1$, furnishes the analytical extension of the function $W(z)$ defined by (2) throughout the strip

$$\lambda_{\omega} < R(z) \leq \lambda.$$

CHAPTER II.

ANALYTICAL EXTENSION OF FUNCTIONS DEFINED BY FACTORIAL SERIES.

I. *Analytical Extension By Use of a Loop Integral.*

4. As has been stated in the Introduction, it is the purpose of this paper to derive new results concerning the analytical extension of functions defined by factorial series. In the results of Nörlund, mentioned in the preceding chapter, the given series is transformed into another factorial series whose line of convergence, in general, lies to the left of the line of convergence, $R(z) = \lambda$, of the given series. Thus, these results furnish a means of obtaining the analytical extension of the function defined by the given series throughout a vertical strip of determinate width lying to the left of the line $R(z) = \lambda$. It is our purpose to obtain expressions which will furnish the analytical extension of such a function over the whole finite z plane, when the coefficients of the given series satisfy certain conditions.

For our present purpose, let us now write series (1) of the Introduction in the form

$$(1) \quad \sum_{n=0}^{\infty} g(n)/z(z+1) \cdots (z+n)$$

where it is understood that z is a variable, and that the coefficients $g(n)$ depend only on n .

We start with a result shown incidentally by Ford in his paper, "On the Behavior of Integral Functions in Distant Portions of the Plane,"* namely that the (entire) function $f(t, z)$ defined by the special series

$$f(t, z) = \sum_{n=0}^{\infty} t^n / \Gamma(z + n + 1)$$

may be written in the following form for any value (real or complex) of t , and for any value of z such that $R(z) > 0$:

$$(2) \quad f(t, z) = \frac{e^t t^{-z}}{\Gamma(z)} \int_0^t e^{-t t^{z-1}} dt.$$

If we restrict ourselves to values of t such that $R(t) > 0$, we may evidently write (2) in the form

$$(3) \quad f(t, z) = \frac{e^t t^{-z}}{\Gamma(z)} \left\{ \int_0^{\infty} e^{-t t^{z-1}} dt - \int_t^{\infty} e^{-t t^{z-1}} dt \right\},$$

where it is understood that in the first integral the integration takes place along the positive axis of reals from $t=0$ to $t=\infty$, while in the second integral it takes place in the direction of the positive real axis from $t=t$ to $t=\infty$. We have now only to recall that

$$\int_0^{\infty} e^{-t t^{z-1}} dt = \Gamma(z)$$

in order to write (3) in the form

$$(4) \quad f(t, z) = e^t t^{-z} \left\{ 1 - \frac{1}{\Gamma(z)} \int_1^{\infty} e^{-t t^{z-1}} dt \right\}$$

But the right hand member here appearing is defined not only when $R(z) > 0$, but is seen to have a meaning whatever be the value of z . Moreover in the neighborhood of any finite point z it is seen to be analytic.

If in particular we take t equal to unity in (4), and multiply both members by $\Gamma(z)$, the product $\Gamma(z) \cdot f(1, z)$ occurring on the left is seen to be the function $G(z)$ defined by the special factorial series

* *Bulletin of American Mathematical Society*, Vol. 34 (1928), p. 91. We here change Ford's notation slightly in order to accommodate his result more aptly to our subsequent discussion.

$$\sum_{n=0}^{\infty} 1/z(z+1) \cdots (z+n).$$

Thus, for the function $G(z)$ defined by this special series, we may write

$$G(z) = e\{\Gamma(z) - \int_1^{\infty} e^{-t} t^{z-1} dt\}.$$

Since the right member of this equation preserves a meaning and is an analytic function of z throughout any region of the z plane which does not include any of the points $z=0, -1, -2, \cdots$, it furnishes the value of $G(z)$ throughout any such region.

We shall now show that this special result may be generalized so as to obtain an analogous one which will apply to factorial series of the first kind in general, at least under some restrictions on the coefficients. In fact, such a generalization is possible as follows:

THEOREM I. *Having given the factorial series*

$$(1) \quad \sum_{n=0}^{\infty} g(n)/z(z+1) \cdots (z+n)$$

let the following power series be formed:

$$g(0) + \frac{\Delta g(0)}{1!}(1-t) + \frac{\Delta^2 g(0)}{2!}(1-t)^2 + \cdots$$

where

$$\Delta g(0) = g(1) - g(0), \quad \Delta^2 g(0) = g(2) - 2g(1) + g(0),$$

etc. Suppose that the function $h(t)$ defined by this series can be expanded about the origin in a series whose radius of convergence exceeds unity. Suppose further that $h(t)$ is analytic throughout an infinite strip of the t plane which shall include the positive real t axis together with the origin, and which may be of arbitrarily small width. Furthermore suppose that the function $h(t)$ is such that the improper integral

$$\int_0^{\infty} h(t) e^{-t} t^{z-1} dt$$

exists, at least for $R(z) > 0$.* Then the function $\Omega(z)$ defined by (1) may be expressed for all values of z in the form

$$(5) \quad \Omega(z) = \frac{e}{e^{2\pi iz} - 1} \int_L h(t) e^{-t} t^{z-1} dt - e \int_1^{\infty} h(t) e^{-t} t^{z-1} dt,$$

* For further comments on these conditions, see Article 5.

with the purpose of determining the function $h(t)$ so that the two series (1) and (9) shall coincide, thus causing the function $\Omega(z)$ defined by (1) to be given by the integral (10). In order that the two series shall coincide, it is necessary and sufficient that

$$\begin{aligned} g(0) &= h(1), \\ g(1) &= h(1) - h'(1), \\ g(2) &= h(1) - 2h'(1) + h''(1), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ g(n) &= \sum_{s=0}^{\infty} (-1)^s \binom{n}{s} h^{(s)}(1), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

These equations are equivalent to the following:

$$\begin{aligned} g(0) &= h(1), \\ \Delta g(0) &= -h'(1), \\ \Delta^2 g(0) &= h''(1), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \Delta^n g(0) &= (-1)^n h^{(n)}(1), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

Hence we must have

$$h(t) = g(0) + \Delta g(0) (1-t) + \Delta^2 g(0) (1-t)^2/2! + \cdots$$

If this function $h(t)$ satisfies the conditions indicated in the hypothesis of the theorem, we can now assert that the function $\Omega(z)$ defined by series (1) may be written in the form

$$(11) \quad \Omega(z) = e \int_0^1 h(t) e^{-t} t^{z-1} dt.$$

provided $R(z) > 0$. If the abscissa of convergence of (1) is greater than zero, then it follows that (11) furnishes the analytical extension of the function $\Omega(z)$ throughout that portion of the z plane lying between the boundary line of convergence and the axis of pure imaginaries.

In order to extend this result so as to obtain the analytical extension of the function $\Omega(z)$ over the whole finite z plane, let us now write the integral appearing in (11) in the form

$$(12) \quad \int_0^1 h(t) e^{-t} t^{z-1} dt = \int_0^{\infty} \bar{h}(t) e^{-t} t^{z-1} dt - \int_1^{\infty} h(t) e^{-t} t^{z-1} dt.$$

By hypothesis, the first integral on the right exists when $R(z) > 0$. When regarded as a function of z this integral is seen to be analytic throughout

any portion of the half-plane $R(z) > 0$, while the second integral is seen at once to be analytic for all values of z . Inasmuch as the first integral may fail to preserve a meaning for $R(z) < 0$, we proceed to study the function $J(z)$ defined by this integral. In this connection let us consider the loop integral

$$(13) \quad l(z) = \int_L h(t) e^{-t} t^{z-1} dt,$$

the contour L extending from infinity along the upper side of the positive real t axis to the point $t = \epsilon$, (ϵ being an arbitrarily small positive number), thence around the circle of radius ϵ about the origin, and returning to infinity along the lower side of the same axis. It will be sufficient for the moment to consider z as real and positive. If we let $t = r(\cos \theta + i \sin \theta)$, then we have $t^{z-1} = \exp [(z-1)(\log r + i\theta)]$, $0 \leq \theta \leq 2\pi$. Along the upper side of the positive real t axis, the angle θ is 0, while along the lower side we have $\theta = 2\pi$. The contribution to $l(z)$ arising from integrating along the upper side of the axis from $t = \infty$ to $t = \epsilon$ is given by the integral

$$- \int_{\epsilon}^{\infty} h(t) e^{-t} t^{z-1} dt$$

while the contribution arising from integrating along the lower side from $t = \epsilon$ to $t = \infty$ is given by

$$e^{2\pi iz} \int_{\epsilon}^{\infty} h(t) e^{-t} t^{z-1} dt.$$

As ϵ approaches zero, the sum of these two contributions approaches as a limit the expression

$$(e^{2\pi iz} - 1)J(z)$$

where

$$J(z) = \int_0^{\infty} h(t) e^{-t} t^{z-1} dt.$$

Upon the circle, we have $t^{z-1} = (\epsilon e^{i\theta})^{z-1}$. Moreover the product $h(t) \cdot e^{-t}$, being analytic at the origin, remains less than a constant k in this neighborhood. Hence the contribution to $l(z)$ arising from integrating around the circle is in absolute value less than

$$k\epsilon^z \left| \int_0^{\pi} e^{iz\theta} d\theta \right| = 2\pi k\epsilon^z.$$

Therefore, as ϵ approaches zero, the contribution in question vanishes. Consequently, we have

$$(14) \quad J(z) = l(z) / (e^{2\pi iz} - 1).$$

Suppose now we let z take on complex values. The left member of (14) is an analytic function of z for $R(z) > 0$. But the right member preserves a meaning and is analytic for all values of z except $z = 0, -1, -2, \dots$. Hence (14) must hold for all values of z except zero and negative integers. Taking account of (12), and (13), we have, finally,

$$(15) \quad \Omega(z) = e/(e^{2\pi iz} - 1) \int_L h(t)e^{-t}t^{z-1}dt - e \int_1^\infty h(t) \cdot e^{-t}t^{z-1}dt$$

Hence the proof of the theorem is complete.

5. *Remarks.* The following remarks relative to the foregoing theorem are now to be noted.

1'. We have required that the function $h(t)$ shall admit an expansion in powers of t which shall have a radius of convergence exceeding unity. If, however, the radius is unity, but the series can be integrated termwise from 0 to 1, then Theorem I still holds provided that the other conditions required of $h(t)$ are satisfied.

2'. We require $h(t)$ to be analytic throughout an infinite strip of arbitrarily small width which shall include the positive real t axis and the origin. This condition will evidently be satisfied by any entire function.

3'. We have required further that the function $h(t)$ be such that the integral

$$\int_0^\infty h(t)e^{-t}t^{z-1}dt$$

shall exist for $R(z) > 0$. If the condition stated in remark 2' is satisfied, then this integral will evidently exist whenever the following statement can be made: Corresponding to any arbitrarily small positive number ϵ there exists a positive constant K_ϵ such that for all values of t sufficiently large, we may write

$$|h(t)| < K_\epsilon e^{t(1-\epsilon)}.$$

We note in this connection that all the conditions imposed on $h(t)$ are satisfied in the special case in which the general coefficient $g(n)$ is a polynomial, for then the function $h(t)$ is also a polynomial.

4'. Nielsen* has shown that if series (1) of the Introduction converges, then the function $\Omega(z)$ defined by that series may be expressed in the form

$$(16) \quad \Omega(z) = \int_0^1 \phi(t)t^{z-1}dt,$$

* (8), pp. 239-241.

where $\phi(t)$ is the function defined by the power series

$$a_0 + a_1(1-t) + a_2(1-t)^2 + \cdots.$$

Moreover Pincherle* has shown that an integral of the type here appearing exists and is an analytic function of z throughout a half-plane $R(z) > \sigma$, where the abscissa σ may be greater than, equal to, or less than the abscissa λ of convergence of the corresponding factorial series.

That the general result obtained in Theorem I gives the particular result of Nielsen when $R(z) > \max(0, \sigma)$ may be shown as follows: We have defined $h(t)$ in the form

$$h(t) = g(0) + \Delta g(0)(1-t) + \Delta^2 g(0)(1-t)^2/2! + \cdots.$$

Furthermore we may write

$$e^{-t} = (1/e) [1 + (1-t) + (1-t)^2/2! + \cdots].$$

Upon multiplying these two series, we obtain

$$\begin{aligned} (17) \quad h(t) \cdot e^{-t} &= (1/e) [g(0) + \{g(0) + \Delta g(0)\}(1-t) \\ &\quad + \{\Delta^2 g(0)/2! + \Delta g(0) + g(0)/2!\}(1-t)^2 + \cdots \\ &\quad + [(\Delta + 1)^n g(0)/n!](1-t)^n + \cdots]. \end{aligned}$$

But

$$\begin{aligned} g(0) + \Delta g(0) &= g(1) = a_1 \\ \Delta^2 g(0)/2! + \Delta g(0) + g(0)/2! &= g(2)/2! = a_2, \end{aligned}$$

and in general

$$(\Delta + 1)^n g(0) = g(n)/n! = a_n.$$

Therefore we may write

$$h(t)e^{-t} = (1/e) [a_0 + a_1(1-t) + a_2(1-t)^2 + \cdots],$$

or

$$h(t)e^{-t} = 1/e \cdot \phi(t).$$

Consequently, for $R(z) > \max(0, \sigma)$, we have

$$(18) \quad e \int_0^1 h(t) e^{-t} t^{z-1} dt = \int_0^1 \phi(t) t^{z-1} dt.$$

Now when $R(z) > 0$, the quantity

$$\frac{1}{e^{2\pi iz} - 1} \int_L h(t) e^{-t} t^{z-1} dt$$

* See Ref. (3).

occurring in the right member of (15) reduces to the integral

$$\int_0^{\infty} h(t) e^{-t} t^{z-1} dt$$

and therefore for $R(z) > 0$ the whole right member reduces at once to the left member of (18). The equivalence of the two results for $R(z) > \max(0, \sigma)$ is thus established. It appears, therefore, that the result of Theorem I goes further than Nielsen's result since the former furnishes the value of the function $\Omega(z)$ throughout the whole finite z plane while the latter furnishes the value of the same function only, in general, for values of z in a certain half-plane.

Examples. 1. Let $\Omega(z)$ be defined by the series

$$\sum_{n=0}^{\infty} n^2/z(z+1) \cdots (z+n).$$

Then we have $h(t) = t^2 - 3t + 2$. Hence we have

$$\Omega(z) = \int_0^1 e^{1-t} (t^2 - 3t + 2) \cdot t^{z-1} dt.$$

This integral fails to have a meaning for $R(z) < 0$, since the integrand becomes infinite at the lower limit. But by Theorem I we have, for any finite z ,

$$\Omega(z) = \frac{e}{e^{2\pi iz} - 1} \int_L (t^2 - 3t + 2) e^{-t} t^{z-1} dt - e \int_1^{\infty} (t^2 - 3t + 2) e^{-t} t^{z-1} dt.$$

2. Consider the series

$$\sum_{n=0}^{\infty} 2^n/z(z+1) \cdots (z+n).$$

Then we have

$$\begin{aligned} h(t) &= 1 + (1-t) + (1/2!)(1-t)^2 + \cdots, \\ &= e^{1-t}. \end{aligned}$$

and therefore we may express $\Omega(z)$ in the form

$$\Omega(z) = e^2 \int_0^1 e^{-2t} t^{z-1} dt.$$

However, this integral has no meaning for $R(z) < 0$, but applying Theorem I we have for any finite value of z ,

$$\Omega(z) = \frac{e^2}{e^{2\pi iz} - 1} \int_L e^{-2t} t^{z-1} dt - e^2 \int_1^{\infty} e^{-2t} t^{z-1} dt.$$

II. *Analytical Extension by the Calculus of Residues.*

6. In this portion of the present chapter we shall employ a method which differs essentially from that used above. In fact we shall employ the calculus of residues to obtain the analytical extension of functions defined by both kinds of factorial series under the assumption that the coefficients alternate in sign, and satisfy certain further conditions.

The calculus of residues has been used by Barnes, Ford, and others in finding the analytical extension of functions defined by various types of power series. As consequences of the results obtained, certain further results pertaining to the asymptotic developments of such functions have been established. The question naturally presents itself as to whether the method mentioned can be used also to obtain analogous results for factorial series. We shall show, in fact, that some such results are possible.

Our investigations are based upon the following fundamental theorem which is a direct consequence of elementary considerations in the theory of functions of a complex variable:*

"If $P(w)$ and $Q(w)$ are any two functions of the complex variable w , both of which are single valued and analytic in a region R of the w complex plane, and of which the latter vanishes at the points $w = \lambda_1, \lambda_2, \dots, \lambda_n$, which are zeroes of the first order, and if C_n denote any contour lying in R and inclosing the points $w = \lambda_1, \lambda_2, \dots, \lambda_n$, then we have

$$(19) \quad \frac{1}{2\pi i} \int_{C_n} \frac{P(w)}{Q(w)} dw = \sum_{t=1}^n \frac{P(\lambda_t)}{Q'(\lambda_t)},$$

where the integration is performed in the positive sense."

Our first theorem is as follows:

THEOREM II. *Given any factorial series of the form*

$$(20) \quad \sum_{n=0}^{\infty} \frac{(-1)^n g(n)}{z(z+1) \cdots (z+n)}$$

whose abscissa λ of convergence is finite. If the function $g(n)$ occurring in the coefficient of this series is such that, when considered as a function $g(w)$ of the complex variable $w = x + iy$, (a) it is single valued and analytic throughout all portions of the w plane lying to the right of (or upon) the vertical line $w = -\frac{1}{2} + iy$, except for a finite number of poles situated at the points $w = \lambda_1, \lambda_2, \dots, \lambda_n$, ($\lambda_t \neq \text{integer}$), and (b) to any arbitrarily small

* See, for example, Ford, "Studies on Divergent Series and Summability," *Michigan Science Series*, Vol. II, 1916, page 184.

positive number ϵ there corresponds a positive constant K_ϵ such that for all values of $x \geq -1/2$ and for all positive values of y sufficiently large we may write

$$\left| \frac{g(x \pm iy)}{g(x)} \right| < K_\epsilon e^{y(\pi/2 - \epsilon)},$$

then the function $\Omega(z)$ defined by series (20) when $R(z) > \lambda$ may be extended analytically throughout the whole finite z plane, except in neighborhoods of the points $z = 0, -1, -2, \dots$, and throughout this region will be defined by the equation

$$(21) \quad \Omega(z) = \frac{\Gamma(z)}{2} \int_{-\infty}^{+\infty} \frac{g(-1/2 + iy) dy}{\Gamma(z + 1/2 + iy) \cosh \pi y} - \sum_{t=1}^n r_t$$

where r_t represents the residue of the function

$$(22) \quad \pi \Gamma(z) g(w) / \Gamma(z + w + 1) \sin \pi w$$

at the point $w = \lambda_t$.

The proof of this theorem is analogous to that of the second part of the general theorem proved by Ford in his paper, "On the Behavior of Integral Functions in Distant Portions of the Plane," already mentioned in the first part of this chapter.

In order to prove the theorem, let us write series (20) in the form

$$(23) \quad \sum_{n=0}^{\infty} (-1)^n \Gamma(z) g(n) / \Gamma(z + n + 1).$$

Furthermore, for simplicity, let us assume first that the function $g(w)$ has no poles to the right of the line $w = -1/2 + iy$. Moreover, let us regard z as having any fixed value which we shall take as real, positive, greater than λ , and not an integer. The theorem then results, as we shall show, from integrating the function (22) about the rectangular contour C_n formed in the w plane by the lines

$$w = -1/2 + iy, \quad w = 1/2 + 2n + iy, \quad w = x \pm ij,$$

where n is any positive integer, and j is an arbitrarily large positive number.

Taking

$$P(w) = \frac{\pi \Gamma(z) g(w)}{\Gamma(z + w + 1)}, \quad Q(w) = \sin \pi w,$$

and applying (19), we arrive at the relation

$$(24) \quad \Gamma(z) \sum_{n=0}^{2n} \frac{(-1)^n g(n)}{\Gamma(z + n + 1)} = \frac{\Gamma(z)}{2i} \int_{C_n} \frac{g(w) dw}{\Gamma(z + w + 1) \sin \pi w}.$$

We now proceed to study the integral here appearing in further detail.

First, along that side of C_n upon which $w = x + ij$, we have $dw = dx$, $\sin \pi w = \sin \pi j (\sin \pi x \cosh \pi j + i \cos \pi x)$. Hence if we denote by I the contribution to the contour integral in (24) arising from integrating along this side, we have

$$I = \frac{\Gamma(z)}{2i \sin h \pi j} \int_{2n+1/2}^{-1/2} \frac{g(x + ij) dx}{\Gamma(z + x + 1 + ij) (\sin \pi x \cosh \pi j + i \cos \pi x)}.$$

Now it is a well known property of the Gamma function that if α and β are real, then we may write *

$$(25) \quad |\Gamma(\alpha + i\beta)| = (2\pi)^{1/2} |\alpha + i\beta|^{\alpha-1/2} e^{-\alpha\beta} \tan^{-1}\beta/\alpha \cdot (1 + \delta)$$

where δ approaches zero as α or β becomes infinite. Moreover as $j \rightarrow +\infty$, $\sinh \pi j$ becomes infinite like $e^{\pi j}$, and when we take account of condition (b) of the hypothesis, it appears that the absolute value of the integrand in I vanishes to as high an order as that of $e^{-\epsilon j} |z + x + 1 + ij|^{z+x+1/2}$ as $j \rightarrow +\infty$. Hence we have at once $\lim_{j \rightarrow \infty} I = 0$.

Similarly, the contribution arising from integrating along that side of C_n upon which $w = x - ij$ will be seen to approach zero as $j \rightarrow +\infty$. For as the expression in question differs from I above only in that j is replaced by $-j$, and as $j \rightarrow +\infty$ the function $\sinh(-\pi j)$ becomes infinite like $e^{\pi j}$, the absolute value of the integrand vanishes like $e^{-\epsilon j} |z + x + 1 - ij|^{z+x+1/2}$ as $j \rightarrow +\infty$.

Next, along that side of C_n along which $w = 2n + 1/2 + iy$, we have $dw = idy$, $\sin \pi w = \cosh \pi y$. If J represents the contribution to the contour integral (24) from the integration along this side, then we have

$$(26) \quad J = \frac{\Gamma(z)}{2} \int_{-\infty}^{+\infty} \frac{g(1/2 + 2n + iy) dy}{\Gamma(z + 2n + 3/2 + iy) \cosh \pi y}.$$

As $y \rightarrow +\infty$, $\cosh \pi y$ becomes infinite like $e^{\pi y}$, while as $y \rightarrow -\infty$ the same function becomes infinite like $e^{-\pi y}$. Upon noting relation (25) and condition (b) of the hypothesis, we see that as $y \rightarrow +\infty$, the absolute value of the integrand vanishes to as high an order as that of $e^{-\epsilon y} |z + 2n + 3/2 + iy|^{z+2n+1}$, while as $y \rightarrow -\infty$, it vanishes to as high an order as that of the quantity $e^{\epsilon y} |z + 2n + 3/2 - iy|^{z+2n+1}$. It follows, therefore, that the improper integral appearing in (26) exists. Moreover we may now show $\lim_{n \rightarrow \infty} J = 0$. To do this, let us write J in the form

* Godefroy, *La Fonction Gamma*, Paris, 1901, page 45.

$$J = \frac{1}{2} \cdot \frac{\Gamma(z) \cdot g(2n)}{\Gamma(z + 2n + 1)} \int_{-\infty}^{+\infty} \frac{g(2n + \frac{1}{2} + iy)}{g(2n)} \cdot \frac{\Gamma(z + 2n + 1)}{\Gamma(z + 2n + \frac{3}{2} + iy)} \cdot \frac{dy}{\cosh \pi y}.$$

Since the given series is assumed convergent for $R(z) > \lambda$, we have at once

$$\lim_{n \rightarrow \infty} \frac{\Gamma(z)g(2n)}{\Gamma(z + 2n + 1)} = 0.$$

Furthermore, from (25), we may write

$$\left| \frac{\Gamma(z + 2n + 1)}{\Gamma(z + 2n + \frac{3}{2} + iy)} \right| < \frac{|z + 2n + \frac{3}{2}|^{z+2n+1}}{|z + 2n + \frac{3}{2} + iy|^{z+2n+1}} e^{-y \tan^{-1}[y/(2n+z+3/2)]} < e^{\pi/2|y|}.$$

Finally, taking account of condition (b) of the hypothesis, we have $\lim J = 0$.

Along the fourth side of C_n , we have $w = -\frac{1}{2} + iy$, $\frac{dw}{dy} = i dy$, $\sin \pi w = -\cosh \pi y$, and the integration takes place from $+\infty$ to $-\infty$. If we now take into consideration the contribution to the contour integral arising from integrating along this side, the fundamental relation (24) takes the following form (n having been allowed to become infinite as indicated above):

$$(27) \quad \Omega(z) = \frac{\Gamma(z)}{2} \int_{-\infty}^{+\infty} \frac{g(-\frac{1}{2} + iy) dy}{\Gamma(z + \frac{1}{2} + iy) \cosh \pi y}.$$

Up to this point we have confined z to values which are real, positive, and greater than λ . Suppose now that z be allowed to take on any value, real or complex. The left member of (27) is an analytic function of z when $R(z) > \lambda$, except in neighborhoods of the points $z = 0, -1, -2, \dots$. We may show, however, that the right member represents an analytic function of z throughout any finite portion T of the z plane which does not — any of the points $z = 0, -1, -2, \dots$. To establish this fact it suffices to show that the improper integral involved converges uniformly throughout any such region. For this purpose, consider the integral

$$(28) \quad \int_1^{\infty} \frac{g(-\frac{1}{2} + iy) dy}{\Gamma(z + \frac{1}{2} + iy) \cosh \pi y}.$$

From (25) and condition (b) of the hypothesis, it appears that the absolute value of the integrand for a sufficiently large y is less than

$$e^{-\epsilon y} |z_1 + \frac{1}{2} + iy|^{z_1}$$

where z_1 denotes the maximum absolute value of z in the proposed region T . But we may write

$$e^{-\epsilon y} |z_1 + \frac{1}{2} + iy|^{z_1} = e^{-(\epsilon/2)y} \cdot e^{-(\epsilon/2)y} |z_1 + \frac{1}{2} + iy|^{z_1} < K e^{-(\epsilon/2)y}$$

where K is a constant. Hence we have

$$\left| \int_l^\infty \frac{g(-\frac{1}{2} + iy) dy}{\Gamma(z + \frac{1}{2} + iy) \cosh \pi y} \right| < K \int_l^\infty e^{-(\epsilon/2)y} dy = (2/\epsilon) K e^{-\epsilon l/2}$$

provided l is a sufficiently large quantity independent of z . Consequently the integral (28) can be made less in absolute value than any arbitrarily small positive number η by proper choice of the lower limit l . In the same way, we can show that

$$\left| \int_{-\infty}^{-l} \frac{g(-\frac{1}{2} + iy) dy}{\Gamma(z + \frac{1}{2} + iy) \cosh \pi y} \right| < \eta$$

provided l is sufficiently large. Thus the uniform convergence of the improper integral in (27) is established. The relation must therefore hold for all values of z except zero and negative integers.

The proof for the special case in which the function $g(w)$ has no poles in the half-plane $R(w) \geq -\frac{1}{2}$ is thus complete. That the theorem is true when we allow $g(w)$ to have poles as indicated in condition (a) of the hypothesis follows when we note that relation (24) continues to hold (n being sufficiently large) provided we add to its left member the expression

$$\sum_{t=0}^n r_t$$

as it is defined in the statement of the theorem.

7. *Remarks.* 1'. The line $w = -\frac{1}{2} + iy$ taken as the left side of the contour C_n can be replaced by any line $w = k + iy$ where $-1 < k < 0$ provided that $g(w)$ is analytic when $R(w) \geq k$, except for a finite number of poles as indicated in condition (a).

2'. Should the given series converge throughout the whole z plane, then the function $\Omega(z)$ is expressed at all points z by (21).

3'. Suppose condition (a) is not satisfied, but the function $g(w)$ has a finite number of branch points situated at the points $w = \lambda_1, \lambda_2, \dots, \lambda_n$. Then (21) continues to hold provided one adds to the left member the sum of the loop integrals at these points. The p -th of these is described as follows: Draw the straight line extending from the point λ_p to infinity in the direction of the positive axis of imaginaries, and let this line be regarded as a cut in the w plane. Now let the loop consist of the two lines drawn on either side of and near the cut, the ends of these lines in the neighborhood

of the point λ_p being joined by a circular arc of small radius drawn about this point. The integration is to be performed in the positive sense.

Examples.

1. Consider the function $\Omega(z)$ defined by the series

$$\Omega(z) = \sum_{n=0}^{\infty} (-1)^n n! / z(z+1) \cdots (z+n).$$

This series converges when $R(z) > 0$, and may be shown to be represented by the integral

$$\int_0^1 [t^{z-1}/(z-t)] dt$$

which in turn preserves a meaning for $R(z) > 0$. Since the function $n!$ or $\Gamma(n+1)$ satisfies both conditions (a) and (b) of the hypothesis of Theorem II, then by that theorem we may write

$$\Omega(z) = \frac{\Gamma(z)}{2} \int_{-\infty}^{+\infty} \frac{\Gamma(1/2 + iy) dy}{\Gamma(z + 1/2 + iy) \cosh \pi y},$$

which holds for all values of z except $0, -1, -2, \dots$.

2. Let $g(n) = h(n) n!$, where the function $h(w)$ satisfies condition (a), while corresponding to any arbitrarily small positive number ϵ there exists a positive constant K_ϵ such that

$$|h(x \pm iy)/h(x)| < K_\epsilon e^{y(\pi - \epsilon)}, \quad x \geq -1/2.$$

then the function $g(w)$ satisfies both conditions (a) and (b), and, assuming that the given series converges, we have

$$\Omega(z) = \frac{\Gamma(z)}{2} \int_{-\infty}^{+\infty} \frac{h(-1/2 + iy) \Gamma(1/2 + iy) dy}{\Gamma(z + 1/2 + iy) \cosh \pi y},$$

this result holding for all finite values of z different from zero and negative integers.

8. So far our results have been concerned with the factorial series of the first kind. We shall now turn our attention to the second kind of factorial series. For this purpose, let us write series (2) of the Introduction in the form

$$g(0) + \sum_{n=1}^{\infty} g(n) (z-1)(z-2) \cdots (z-n)$$

where, as above, z denotes a variable, and where the general coefficient $g(n)$ depends only on n . Then we have the following theorem which is analogous to Theorem II:

THEOREM III. Given any factorial series of the form

$$(29) \quad g(0) + \sum_{n=1}^{\infty} (-1)^n g(n) (z-1)(z-2) \cdots (z-n),$$

whose abscissa of convergence is finite. If the function $g(n)$ occurring in the general coefficient of this series is such that when considered as a function $g(w)$ of the complex variable $w = x + iy$ (a) it is single-valued and analytic throughout all portions of the half-plane $R(w) \geq -\frac{1}{2}$, except for a finite number of poles situated at the points $w = \lambda_1, \lambda_2, \cdots, \lambda_n$, none of which are integers, and (b) to any arbitrarily small positive number ϵ there corresponds a positive constant K_ϵ such that for $x \geq -\frac{1}{2}$ and for all positive values of y sufficiently large we may write

$$\left| \frac{g(x \pm iy)}{g(x)} \right| < K_\epsilon e^{y(\pi/2 - \epsilon)}$$

then the function $W(z)$ defined by series (29) may be extended analytically throughout all finite portions of the z plane except in the neighborhoods of the points $z = 0, -1, -2, \cdots$, and throughout such regions will be defined by the equation

$$W(z) = \frac{\Gamma(z)}{2} \int_{-\infty}^{+\infty} \frac{g(-\frac{1}{2} + iy) dy}{\Gamma(z + \frac{1}{2} - iy) \cosh \pi y} - \sum_{t=1}^n r_t$$

where r_t represents the residue of the function

$$\frac{\pi g(w) \cdot \Gamma(z)}{\Gamma(z-w) \sin \pi w}$$

at the point $w = \lambda_t$.

In fact the given series may be written in the form

$$\sum_{n=0}^{\infty} (-1)^n g(n) \Gamma(z) / \Gamma(z-n)$$

and the proof of the theorem is analogous to the proof of Theorem I above. It will therefore be left to the reader.

9. If the function $g(w)$ is single valued and analytic throughout the half-plane $R(w) \geq -q - \frac{1}{2}$, where q is any positive integer, then by taking the line $w = -q - \frac{1}{2} + iy$ as the left side of the contour C_n used in the proofs of the last two theorems, we may show that the following equation holds:

$$(30) \quad W(z) = \sum_{n=-1}^{-q} \frac{(-1)^n g(n) \Gamma(z)}{\Gamma(z-n)} + \frac{(-1)^q \Gamma(z)}{2} \int_{-\infty}^{+\infty} \frac{g(-q - \frac{1}{2} + iy) dy}{\Gamma(z + q + \frac{1}{2} - iy) \cosh \pi y}.$$

If $g(w)$ has a finite number of poles situated at the points $w = \lambda_1, \lambda_2, \dots, \lambda_n$, none of which are integers, then we must add to the left member of (30) the sum of the residues of the function

$$\pi g(w) \Gamma(z) / \Gamma(z - w) \sin \pi w$$

at the points λ_i .

As a consequence of (30), we have the following

COROLLARY. If the coefficient $g(n)$ occurring in the general coefficient of series (29), when considered as a function of the complex variable $w = x + iy$, is such that (a) it is single-valued and analytic throughout the finite w plane, except for a finite number of poles $\lambda_1, \lambda_2, \dots, \lambda_n$, none of which is an integer, and (b) to any arbitrarily small positive number ϵ there corresponds a positive constant K_ϵ such that for all values of x and for all values of y sufficiently large we may write

$$\left| \frac{g(x \pm iy)}{g(x)} \right| < K_\epsilon e^{y(\pi/2 - \epsilon)},$$

then the function $W(z)$ defined by (29) will be such that for all values of z lying in any sector (vertex at $z = 0$) which does not include the negative real axis, we may write

$$W(z) \sim - \sum_{t=1}^n r_t + \frac{g(-1)}{z} - \frac{g(-2)}{z(z+1)} + \frac{g(-3)}{z(z+1)(z+2)} - \dots$$

In fact the expression

$$(31) \quad \Gamma(z) \int_{-\infty}^{+\infty} \frac{g(-q - \frac{1}{2} + iy) dy}{\Gamma(z + q + \frac{1}{2} - iy) \cosh \pi y}$$

may be written in the form

$$\frac{\Gamma(z)}{\Gamma(z + q + \frac{1}{2})} \int_{-\infty}^{+\infty} \frac{g(-q - \frac{1}{2} + iy) \Gamma(z + q + \frac{1}{2}) dy}{\Gamma(z + q + \frac{1}{2} - iy) \cosh \pi y}$$

or

$$\frac{\Gamma(z)}{\Gamma(z + q + \frac{1}{2})} \int_{-\infty}^{+\infty} \frac{g(-q - \frac{1}{2} + iy) B(z + q - \frac{1}{2}, 1 - iy) dy}{\Gamma(z - iy) \cosh \pi y}$$

where B is the customary symbol for the Beta function. Now we have from the definition of the Beta function

$$\begin{aligned} |B(z + q - \frac{1}{2}, 1 - iy)| &< \left| \int_0^1 x^{z+q-3/2} (1-x)^{-iy} dx \right| \\ &< \left| \int_0^1 x^{z+q-3/2} dx \right| = 1/|z + q - \frac{1}{2}|. \end{aligned}$$

Recalling that

$$(z + q - \tfrac{1}{2})\Gamma(z + q - \tfrac{1}{2}) = \Gamma(z + q + \tfrac{1}{2}),$$

we see that the absolute value of the remainder (31) is less than

$$\left| \frac{\Gamma(z)}{\Gamma(z + q + \tfrac{1}{2})} \right| \cdot \int_{-\infty}^{+\infty} \frac{|g(-q - \tfrac{1}{2} + iy)|}{|\Gamma(1 - iy)| \cosh \pi y} dy.$$

By virtue of (25) and condition (b) of the hypothesis, the integral here appearing exists. Furthermore, by (25), the expression

$$\left| \frac{\Gamma(z)}{\Gamma(z + q + \tfrac{1}{2})} \right|$$

vanishes to as high an order as the $(q - \tfrac{1}{2}) - th$ as $|z| \rightarrow \infty$, provided that z does not go to infinity along the negative real axis. Hence the corollary follows.

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